

Centred quadratic stochastic operators

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Abstract

We study the weak convergence of iterates of so-called centred kernel quadratic stochastic operators. These iterations, in a population evolution setting, describe the additive perturbation of the arithmetic mean of the traits of an individual's parents and correspond to certain weighted sums of independent random variables. We show that one can obtain weak convergence results under rather mild assumptions on the kernel. Essentially it is sufficient for the distribution of the perturbing random variable to have a finite variance or have tails controlled by a power function. The advantage of these conditions is that in many cases they are easily verifiable by an applied user. Additionally, the representation by sums of random variables implies an efficient simulation algorithm to obtain random variables approximately following the law of the iterates of the quadratic stochastic operator, with full control of the degree of approximation. Our results also indicate where lies an intrinsic difficulty in the analysis of the behaviour of quadratic stochastic operators.

Keywords : Asymptotic stability, Nonlinear Markov process, Phenotypic evolution, Quadratic stochastic operators, Simulation, Weak convergence

1 Introduction

The theory of quadratic stochastic operators (QSOs) is rooted in the work of Bernstein (1924). He applied such operators to model the evolution of a discrete probability distribution of a finite number of biotypes in a process of inheritance. The problem of a description of their trajectories was stated by Ulam (1960). Since the seventies of the 20th century limiting behaviour of iterates of quadratic

stochastic operators was intensively studied (see e.g. Kesten, 1970; Lyubich, 1971; Vallander, 1972; Zakharevich, 1978; Ganikhodzhaev and Zanin, 2004; Barański and Misiurewicz, 2010). The field is steadily evolving in many directions (see Ganikhodzhaev et al., 2011, for a detailed review of mathematical results and open problems). Recently Bartoszek and Pułka (2013b) introduced and examined in detail different types of asymptotic behaviours of quadratic stochastic operators in the (discrete) ℓ^1 case. The results obtained there were subsequently generalized to the (continuous) L^1 case by Bartoszek and Pułka (2015a,b). Furthermore, Bartoszek and Pułka (2013a) described an algorithm to simulate the behaviour of iterates of quadratic stochastic operators acting on the ℓ^1 space. However, it should be stressed that direct applications of quadratic stochastic operators are still in their infancy even in a discrete case. Currently Ganikhodzhaev et al. (2004), Ganikhodzhaev et al. (2010) and Ganikhodzhaev et al. (2013) can serve as notable examples, which also illustrate the complexity of the concerned problem. If one now starts to consider QSOs acting on L^1 then the situation becomes even more complicated, in a sense because Schur’s lemma does not hold. To obtain results one needs to make restrictive assumptions on the QSO, e.g. Bartoszek and Pułka (2015a) assume a kernel form (Definition 2.3). But even in this subclass it is not readily possible to prove convergence of a trajectory of a QSO. Very recently Rudnicki and Zwoleński (2015) and Zwoleński (2015) considered an even more restrictive subclass of kernel QSOs, that correspond to a model which “retains the mean” (according to Eq. (9) by Rudnicki and Zwoleński, 2015). With these (and additional technical assumptions, like bounds on moment growth) they were able to obtain a convergence of the iterates, which is slightly stronger than the weak convergence. Here, motivated by the model described in Rudnicki and Zwoleński (2015)’s Example 1 we consider a very special but biologically extremely relevant type of “mean retention” where the kernel of the QSO corresponds to an additive perturbation of the parents’ traits (comp. Definition 3.1). This is of course less general than Rudnicki and Zwoleński (2015)’s Theorems 3 and 4, and substantially different than the particular case of their Eq. (9). First, we consider discrete time evolution. Moreover, we are concerned with weak convergence only. But it is the price we pay for being allowed to drop the assumptions of kernel continuity, moment growth, technical bounds on elements of the death process and other elements of the continuous time process’ generator and kernel. Instead, we need for the perturbing term either a finite second moment or control of the tails of its distribution by a power function, (Theorems 4.1 and 4.3). This does unfortunately also result in the loss of uniqueness of the limit (cf. Rudnicki and Zwoleński, 2015; Zwoleński, 2015) — it is seed specific. On the one hand this

might seem to a very serious drawback, certainly a global attractor is a more desirable result. But on the other hand there are numerous situations, e.g. computer simulations of a system, where one is first interested if a system stabilizes when started from any initial condition, not necessarily at the same state (e.g. the classical Lotka–Volterra system does not have a unique limiting cycle). Furthermore the conditions we provide are very easy to verify, something which is desirable for an applied scientist. More importantly the representation of the iterates of the QSO described by Eq. (7) allows one to implement an efficient simulation algorithm and possibly obtain convergence rates for a given QSO under study.

2 Preliminaries

Let (X, \mathcal{A}) be a separable measure space. By $\mathcal{M} = \mathcal{M}(X, \mathcal{A}, \|F\|_{TV})$ we denote the Banach lattice of all signed measures on X with finite total variation where the norm is given by

$$\|F\|_{TV} := \sup_{f \in X} \{ |\langle F, f \rangle| : f \text{ is } \mathcal{A} - \text{measurable}, \sup_{x \in X} |f(x)| \leq 1 \},$$

where

$$\langle F, f \rangle := \int_X f(x) dF(x).$$

By $\mathcal{P} := \mathcal{P}(X, \mathcal{A})$ we denote the convex set of all probability measures on (X, \mathcal{A}) . Spaces constructed as the above \mathcal{M} , or appropriate subspaces of such \mathcal{M} , can serve as the state-spaces of processes describing the evolution of probability distributions of X -valued traits of some species of interest. This evolution maybe governed for example by the below concept of a quadratic stochastic operator, extending Bartoszek and Pułka (2015a)'s definition. Let \mathcal{M}_0 be a Banach subspace on \mathcal{M} , and let $\mathcal{P}_0 := \mathcal{P} \cap \mathcal{M}_0$.

Definition 2.1 *A bilinear symmetric operator $\mathbf{Q}: \mathcal{M}_0 \times \mathcal{M}_0 \rightarrow \mathcal{M}_0$ is called a quadratic stochastic operator on \mathcal{M}_0 if*

$$\|\mathbf{Q}(F_1, F_2)\|_{TV} = \|F_1\|_{TV} \|F_2\|_{TV} \text{ for all } F_1, F_2 \in \mathcal{M}_0, \text{ and } \mathbf{Q}(F_1, F_2) \geq 0, \text{ if } F_1, F_2 \geq 0.$$

Notice that QSOs are bounded as $\sup_{\|F_1\|_{TV}=1, \|F_2\|_{TV}=1} \|\mathbf{Q}(F_1, F_2)\|_{TV} = 1$. Moreover, if $\tilde{F} \geq F \geq 0$ and $\tilde{G} \geq G \geq 0$ then $\mathbf{Q}(\tilde{F}, \tilde{G}) \geq \mathbf{Q}(F, G)$. Clearly, $\mathbf{Q}(\mathcal{P}_0 \times$

$\mathcal{P}_0) \subseteq \mathcal{P}_0$. Such QSOs have an interpretation in evolutionary biology. Namely, imagine that we observe two populations, where $F_1, F_2 \in \mathcal{P}_0$ represent their trait distributions. Then $\mathbf{Q}(F_1, F_2) \in \mathcal{P}_0$ represents a distribution of this trait in the next generation coming from the mating of independent individuals from two different populations. Special attention is paid to the nonlinear “diagonalized” mapping $\mathcal{P}_0 \ni F \mapsto \mathbb{Q}(F) := \mathbf{Q}(F, F) \in \mathcal{P}_0$. Then the values of the sequence of iterates $\mathbb{Q}^n(F)$, $n = 0, 1, 2, \dots$, model the evolution of the probability distribution of the X -valued trait of an inbreeding or hermaphroditic population, with F as the initial distribution. Hence a typical question when working with quadratic stochastic operators is their long-term behaviour.

Let us reformulate the different types of asymptotic behaviour of quadratic stochastic operators, originally considered by Bartoszek and Pułka (2013b, 2015a).

Definition 2.2 *A quadratic stochastic operator \mathbf{Q} on the Banach subspace \mathcal{M}_0 is called:*

1. *norm mixing (also called uniformly asymptotically stable) if there exists a probability measure $H \in \mathcal{P}_0$ such that*

$$\lim_{n \rightarrow \infty} \sup_{F \in \mathcal{P}_0} \|\mathbb{Q}^n(F) - H\|_{TV} = 0 ,$$

2. *strong mixing (asymptotically stable) if there exists a probability measure $H \in \mathcal{P}_0$ such that for all $F \in \mathcal{P}_0$ we have*

$$\lim_{n \rightarrow \infty} \|\mathbb{Q}^n(F) - H\|_{TV} = 0 ,$$

3. *strong almost mixing if for all $F_1, F_2 \in \mathcal{P}_0$ we have*

$$\lim_{n \rightarrow \infty} \|\mathbb{Q}^n(F_1) - \mathbb{Q}^n(F_2)\|_{TV} = 0 .$$

Bartoszek and Pułka (2015a) distinguished the kernel subclass of quadratic stochastic operators originally defined on the Banach lattice $L^1(\mu) = L^1(X, \mathcal{A}, \mu)$ of absolutely integrable real valued functions with respect to a fixed σ -finite positive measure μ . However $L^1(\mu)$ is isometrically isomorphic to the subspace $\mathcal{M}(\mu) := \mathcal{M}(X, \mathcal{A}, \mu)$ of measures from \mathcal{M} absolutely continuous with respect to μ . Therefore through the equality $L^1 \ni f \rightarrow (F(A) = \int_A f \, d\mu : A \in \mathcal{A}) \in \mathcal{M}(\mu)$, they may be equivalently defined as below.

Definition 2.3 A quadratic stochastic operator $\mathbf{Q}: \mathcal{M}(\mu) \times \mathcal{M}(\mu) \rightarrow \mathcal{M}(\mu)$ is called a kernel quadratic stochastic operator if there exists an $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ -measurable, nonnegative function $q: X \times X \times X \rightarrow \mathbb{R}_+$, such that $q(x, y, z) = q(y, x, z)$ for any $x, y, z \in X$, $\int_X q(x, y, z) d\mu(z) = 1$ for $\mu \times \mu$ -almost all $(x, y) \in X \times X$, and $\mathbf{Q}(F_1, F_2) = \mathbf{Q}_q(F_1, F_2)$, where

$$\mathbf{Q}_q(F_1, F_2)(A) = \int_A \int_X \int_X f_1(x) f_2(y) q(x, y, z) d\mu(x) d\mu(y) d\mu(z), \quad A \in \mathcal{A},$$

for measures F_i with densities f_i , $i = 1, 2$.

Bartoszek and Pułka (2015a) provide a detailed study of the limit behaviour of (kernel) quadratic stochastic operators. In particular, equivalent conditions for (uniform) asymptotic stability of such operators are expressed in terms of non-homogeneous chains of linear Markov operators. There are of course other relevant works in the literature dealing with the topic of limit behaviour of quadratic stochastic operators. For instance, Ganikhodjaev and Hamzah (2014); Ganikhodjaev et al. (2014a,b) recently studied non-ergodicity of QSOs.

Many models do not require the strong convergence of the considered trait distributions. Weak convergence, especially for vector valued traits or for those concentrated on finite sets, seems perfectly sufficient. Therefore we introduce another type of long-term behaviour of quadratic stochastic operators based on the weak convergence of measures. With this in mind in what follows we make the below crucial assumption about \mathcal{M} .

Assumption 2.4 \mathcal{M} is the Banach lattice of finite Borel measures on the trait value space $\langle X, \mathcal{A} \rangle$ equal to a complete separable metric space with \mathcal{A} consisting of Borel sets (generated by the open subsets of X).

Then a sequence of measures $F_n \in \mathcal{M}$ is said to be *weakly convergent* to a measure $H \in \mathcal{M}$, if for every continuous bounded function $f \in C(X)$ the functionals $\langle F_n, f \rangle$ approach $\langle H, f \rangle$, as $n \rightarrow \infty$ (cf. Billingsley, 1979).

Definition 2.5 The quadratic stochastic operator \mathbf{Q} on \mathcal{M} is said to be *weakly asymptotically stable* at $F \in \mathcal{P}$ if there is an $H \in \mathcal{P}$, such that the sequence of probability measures $\mathbf{Q}^n(F)_{n \in \mathbb{N}}$ converges weakly to H .

In the next sections we study in details the situation where the trait values belong to finite dimensional real vector space. This natural setting allows us to exploit the apparatus of characteristic functions.

3 The centred QSO in \mathbb{R}^d

We will focus on a very specific subclass of quadratic stochastic operators which we call *centred*. For this we assume $X = \mathbb{R}^d$, $d \in \mathbb{N}_+$, for the trait value space. Thus the state space equals the lattice $\mathcal{M}^{(d)} = \mathcal{M}(\mathbb{R}^d, \mathcal{B}^{(d)})$ of all finite Borel measures on \mathbb{R}^d with finite variation. The corresponding probability distributions form a convex subset denoted by $\mathcal{P}^{(d)} = \mathcal{P}(\mathbb{R}^d, \mathcal{B}^{(d)})$. The convolution of elements of $\mathcal{M}^{(d)}$ is defined as

$$F * G(A) := \int_{\mathbb{R}^d} F(A - y) \, dG(y), \quad A \in \mathcal{B}^{(d)}.$$

For any $F \in \mathcal{M}^{(d)}$ and $n \in \mathbb{N}_1$ we write $F^{*n} := \underbrace{F * F * \dots * F}_{n \text{ factors}}$ for the n -th convolutive power of F . Furthermore, we write $\dot{F}(A) := F(2 \cdot A)$, $A \in \mathbb{R}^d$.

Definition 3.1 Let $G \in \mathcal{P}^{(d)}$. The associated with G operator $\mathbf{Q}_G: \mathcal{M}^{(d)} \times \mathcal{M}^{(d)} \rightarrow \mathcal{M}^{(d)}$ defined by

$$\mathbf{Q}_G(F_1, F_2) := \dot{F}_1 * \dot{F}_2 * G, \quad (1)$$

is called a centred quadratic stochastic operator. If additionally G is absolutely continuous with respect to the Lebesgue measure $\lambda^{(d)}$, then \mathbf{Q}_G is called a centred kernel quadratic stochastic operator.

We omit the straightforward proof, that the above defined operator \mathbf{Q}_G is a QSO. The following example briefly explains why for absolutely continuous measures G the name *kernel* introduced in Definition 3.1 is applicable.

Example 3.2 If F_1, F_2, G are probability measures on \mathbb{R}^d absolutely continuous with respect to the Lebesgue measure $\lambda^{(d)}$ then their densities $f_1 := \frac{dF_1}{d\lambda^{(d)}}$, $f_2 := \frac{dF_2}{d\lambda^{(d)}}$, $g := \frac{dG}{d\lambda^{(d)}}$ are elements of $L^1 = L^1(\mathbb{R}^{(d)}, \mathcal{B}^{(d)}, \lambda^{(d)})$ and we may write

$$\frac{d}{d\lambda^{(d)}} \mathbf{Q}_G(F_1, F_2)(z) = \int_{\mathbb{R}} \int_{\mathbb{R}} f_1(x) f_2(y) g(z - \frac{x+y}{2}) \, dx \, dy, \quad z \in \mathbb{R}.$$

Indeed, denoting for any $f, h \in L^1$ their (density-type) convolution by

$$f \circledast h(z) := \int_{\mathbb{R}^d} f(z - y) h(y) \, dy,$$

for $z \in \mathbb{R}^d$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f_1(x) f_2(y) g(z - \frac{x+y}{2}) dx dy &= \int_{\mathbb{R}^d} f_1(x) \left(\int_{\mathbb{R}^d} \dot{f}_2(y) g((z - \frac{x}{2}) - y) dy \right) dx \\
&= \int_{\mathbb{R}} f_1(x) (\dot{f}_2 \otimes g(z - \frac{x}{2})) dx \\
&= \dot{f}_1 \otimes \dot{f}_2 \otimes g(z) \\
&= \frac{d}{d\lambda^{(d)}} \mathbf{Q}_G(F_1, F_2)(z),
\end{aligned}$$

since $\dot{f}_i(x) := \frac{d\dot{F}_i}{d\lambda^{(d)}}(x) = 2f_i(2 \cdot x)$. Thus, according to Definition 2.3, \mathbf{Q}_G equals the kernel quadratic stochastic operator \mathbf{Q}_q on $L^1(\lambda^{(d)})$ with $q(x, y, z) = g(z - \frac{x+y}{2})$, $x, y, z \in \mathbb{R}^d$.

As before, we pay special attention to the corresponding “diagonalized” mapping

$$\mathcal{M}^{(d)} \ni F \mapsto \mathbb{Q}_G(F) := \mathbf{Q}_G(F, F) \in \mathcal{M}^{(d)}, \quad (2)$$

where $G \in \mathcal{P}^{(d)}$ is arbitrarily fixed. For a given $F \in \mathcal{M}^{(d)}$ and natural number n we denote the result of the n -th iterate by

$$H^{(n)} := (\mathbb{Q}_G)^n(F). \quad (3)$$

For any $H \in \mathcal{M}^{(d)}$ we define its characteristic function by

$$\varphi_H(s) := \int_{\mathbb{R}^d} \exp(is \cdot x) dH(x), \quad s \in \mathbb{R}^d,$$

where \cdot stands for the canonical scalar product in \mathbb{R}^d . In these terms we have

$$\varphi_{H^{(n)}}(s) = \left(\varphi_F \left(\frac{s}{2^n} \right) \right)^{2^n} \prod_{j=0}^{n-1} \left(\varphi_G \left(\frac{s}{2^j} \right) \right)^{2^j}, \quad s \in \mathbb{R}, n \in \mathbb{N}_+. \quad (4)$$

Indeed, first notice that

$$\varphi_{H^{(1)}}(s) = \varphi_{\mathbb{Q}_G(F)}(s) = \varphi_{\dot{F}}(s) \cdot \varphi_{\dot{F}}(s) \cdot \varphi_G(s) = \left(\varphi_F \left(\frac{s}{2} \right) \right)^2 \cdot \varphi_G(s).$$

Similarly, for $m := n + 1$ from the n -th equation we get

$$\begin{aligned}
\varphi_{H^{\otimes n}}(s) &= \varphi_{\mathbb{Q}_G(H^{\otimes n})}(s) = \left(\varphi_{H^{\otimes 2}}\left(\frac{s}{2}\right) \right)^2 \cdot \varphi_G(s) \\
&= \dots = \left(\varphi_F\left(\frac{s/2}{2^n}\right) \right)^{2^n \cdot 2} \cdot \prod_{j=0}^{n-1} \left(\varphi_G\left(\frac{s/2}{2^j}\right) \right)^{2^j \cdot 2} \cdot \varphi_G(s) \\
&= \left(\varphi_F\left(\frac{s}{2^{n+1}}\right) \right)^{2^{n+1}} \cdot \prod_{j=0}^n \left(\varphi_G\left(\frac{s}{2^j}\right) \right)^{2^j}.
\end{aligned}$$

Thus, by induction Eq. (4) holds for all natural n . Consequently, by the Lévy-Cramér continuity theorem (Theorem 3.1, Chapter 13 Shorack, 2000) and paraphrasing Definition 2.5 we may say that for $G \in \mathcal{P}^{(d)}$ a centred quadratic stochastic operator \mathbf{Q}_G is weakly asymptotically stable at $F \in \mathcal{P}^{(d)}$ if for some $H \in \mathcal{P}^{(d)}$ and for every $s \in \mathbb{R}^d$ the characteristic function $\varphi_{H^{\otimes n}}(s)$ approaches $\varphi_H(s)$ as $n \rightarrow \infty$.

As we show below, the dependence on the distribution of F is substantial. However equality of the limit for different initial distributions can be achieved when F belongs to suitable subclasses. An exemplary subclass consists of distributions with common finite mean value (whenever the limit exists for at least one of its members).

4 Main results

Let us fix $F, G \in \mathcal{P}^{(d)}$. The QSO \mathbb{Q}_G , acting on $(\mathbb{R}^d, \mathcal{B}^{(d)})$, is defined by Eq. (2), and the values of its iterates are given by Eq. (3). Moreover, for any probability distribution $H \in \mathcal{P}^{(d)}$, φ_H denotes its characteristic function, and the vector of moments of order 1 and the covariance matrix are defined by

$$\begin{aligned}
m_H^{(1)} &:= \int_{\mathbb{R}^d} x \, dH(x) = \left[\int_{\mathbb{R}^d} x_j \, dH(x) : j \in \{1, 2, \dots, d\} \right], \\
v_H &:= \left[\int_{\mathbb{R}^d} x_j x_k \, dH(x) : (j, k) \in \{1, 2, \dots, d\}^2 \right],
\end{aligned}$$

whenever they exist in \mathbb{R}^d and $\mathbb{R}^{d \times d}$, respectively.

Theorem 4.1 *Let $F \in \mathcal{P}^{(d)}$ have finite first moments $m := m_F^{(1)} \in \mathbb{R}^d$ and let all first and second moments of $G \in \mathcal{P}^{(d)}$ be finite. We set $m_G^{(1)} = 0 \in \mathbb{R}^d$, and*

$v := v_G \in \mathbb{R}^{d \times d}$. Then \mathbb{Q}_G is weakly stable at F or more precisely, the sequence $(H^{(n)})_{n \in \mathbb{N}} \subset \mathcal{P}^{(d)}$ converges weakly to $H^{(\infty)} \in \mathcal{P}^{(d)}$ with characteristic function equal to

$$\varphi_{H^{(\infty)}}(s) = e^{im \cdot s} \varphi_{G^{(\infty)}}(s), \text{ where } \varphi_{G^{(\infty)}}(s) := \lim_{n \rightarrow \infty} \varphi_{G^{(n)}}(s), \quad s \in \mathbb{R}^d, \quad (5)$$

$$\varphi_{G^{(n)}}(s) := \prod_{j=0}^{n-1} \left(\varphi_G \left(\frac{s}{2^j} \right) \right)^{2^j}, \quad s \in \mathbb{R}^d. \quad (6)$$

PROOF According to Eq. (4) for any natural number n , $\varphi_{H^{(n)}}(s)$ is a characteristic function of the random d -dimensional vector $Z^{(n)} := X^{(n)} + Y^{(n)}$, where

$$\begin{aligned} X^{(n)} &:= \frac{X_1 + X_2 + \dots + X_{2^n}}{2^n}, \\ Y^{(n)} &:= \sum_{j=0}^{n-1} \frac{Y_1^{(j)} + Y_2^{(j)} + \dots + Y_{2^j}^{(j)}}{2^j} \end{aligned} \quad (7)$$

and X_1, X_2, X_3, \dots and $Y_1^{(0)}, Y_1^{(1)}, Y_2^{(1)}, Y_1^{(2)}, \dots, Y_4^{(2)}, \dots, Y_1^{(j)}, \dots, Y_{2^j}^{(j)}, \dots$ are independent sequences of random vectors such that X_1, X_2, X_3, \dots are independent identically distributed according to F and $Y_1^{(0)}, Y_1^{(1)}, Y_2^{(1)}, Y_1^{(2)}, \dots, Y_4^{(2)}, \dots, Y_1^{(j)}, \dots, Y_{2^j}^{(j)}, \dots$ are independent identically distributed according to G . Since $m_F^{(1)} =: m \in \mathbb{R}^d$, by the Strong Law of Large Numbers we obtain that $\lim_{n \rightarrow \infty} X^{(n)} = m$ almost surely. Hence for the first factor of (4) we have

$$\lim_{n \rightarrow \infty} \left(\varphi_F \left(\frac{s}{2^n} \right) \right)^{2^n} = e^{im \cdot s}.$$

The assumptions taken on G imply that the covariance matrix of the independent random vectors

$U_j := (Y_1^{(j)} + Y_2^{(j)} + \dots + Y_{2^j}^{(j)})/2^j$, $j = 0, 1, 2, \dots$, equals $v/2^j$ for any $j = 0, 1, 2, \dots$, and hence the series $Y^{\odot} := \sum_{j=0}^{\infty} U_j$ converges almost surely (as it converges coordinatewise, cf. Durrett, 2010, Theorem 2.5.3). Thus, the probability distribution $G^{(n)}$ of $Y^{(n)}$ converges weakly to the probability distribution of Y^{\odot} . Therefore, again by the continuity theorem, the limit $\varphi_{G^{\odot}}(s)$ of the second factor of (4) is the characteristic function of the probability distribution of Y^{\odot} . By all of the above we obtain that $\varphi_{H^{(n)}}(s) \rightarrow e^{im \cdot s} \varphi_{G^{\odot}}(s)$ for every $s \in \mathbb{R}$, where the limiting function is the characteristic function of the probability distribution of $m + Y^{\odot}$.

□

Remark 4.2 *We can observe that after a large number of iterations the starting distribution is only responsible for the expectation of the law of $\mathbb{Q}_{H^{(n)}}(\cdot)$. It would be tempting to suspect a central limit theorem will hold for kernel part described by $Y^{(n)}$. However this will not occur as U_j tends almost surely to 0. Or in other words the tail elements of the product defining $\varphi_{G^{\odot}}(s)$ tend to 1. This means that the limiting distribution of $Y^{(n)}$ essentially depends on the initial elements of the sequence $\{U_j\}$ and not on the tail ones “normalizing everything” as is the case for CLTs. This makes it difficult to make closed form statements about the law of G^{\odot} .*

For further analysis we confine ourselves to the one dimensional case ($d = 1$). Due to the factorization of the characteristic function of $\varphi_{H^{(n)}}$ expressed by Eq. (4), the sufficient conditions for stability are divided into two steps – first the existence of the limit probability distributions of the arithmetic means $X^{(n)}$ and second — of the weighted sums $Y^{(n)}$, both given by Eq. (7). For the former, we apply the well known theory of stable probability distributions (see e.g. Chapter 17 and Defn. 2.2 in Chapter 15 of Feller, 1966; Shorack, 2000, respectively). Accordingly we can state the following.

Theorem 4.3

(i) *If the weak limit F^{\odot} of the probability distributions $F^{(n)}$ of $X^{(n)}$ exists in $\mathcal{P}^{(1)}$, then the limiting characteristic function $\varphi_{F^{\odot}}$ satisfies the equation*

$$\varphi_{F^{\odot}}(2s) = \left(\varphi_{F^{\odot}}(s) \right)^2, \text{ for all } s \in \mathbb{R}.$$

(ii) If the one-dimensional probability measure $F \in \mathcal{P}^{(1)}$ belongs to the strict domain of attraction of a stable probability measure with characteristic exponent 1, i.e. if for some $S \in \mathcal{P}^{(1)}$ we have $(\varphi_{F^{*n}}(\frac{s}{n}))^n \rightarrow \varphi_S(s)$, as $n \rightarrow \infty$, for $s \in \mathbb{R}$, then the weak limit distribution F^{\odot} of the averages $X^{(n)}$ equals S , too, which is a Cauchy probability distribution, i.e. for some $c \geq 0$, $m \in \mathbb{R}$

$$\varphi_{F^{\odot}}(s) = \exp\{-c|s| + ims\}, \quad s \in \mathbb{R}.$$

(iii) Let the one-dimensional probability distribution $G \in \mathcal{P}^{(1)}$ satisfy the following condition

$$|\ln \varphi_G(s)| \leq A|s|^p \text{ for any } |s| \leq s_0 \text{ for some reals } s_0 > 0, p > 1, A > 0.$$

Then $(G^{(n)})_{n \in \mathbb{N}}$ converges weakly to a probability distribution $G^{\odot} \in \mathcal{P}^{(1)}$ whose characteristic function is given by the infinite product of Eq. (5).

(iv) Under the conditions of (i) and (iii) $(H^{(n)})_{n \in \mathbb{N}}$ converges weakly to $F^{\odot} * G^{\odot}$.

PROOF For the first claim, let $n \rightarrow \infty$. Then

$$\varphi_{F^{\odot}}(2s) = \lim_{n \rightarrow \infty} \varphi_F\left(\frac{2s}{2^n}\right)^{2^n} = \lim_{n \rightarrow \infty} \left(\varphi_F\left(\frac{s}{2^{n-1}}\right)^{2^{n-1}}\right)^2 = \left(\varphi_{F^{\odot}}(s)\right)^2.$$

By the assumptions of (ii), equality $\varphi_{F^{\odot}} = S$ is obvious. Moreover, the equalities follow: $\varphi_S(ns) = (\varphi_S(s))^n$, for all $n \in \mathbb{N}$, $s \in \mathbb{R}$. Thus the claim is a well known result on stable distributions with characteristic exponent 1, (cf. Theorem 3.1 in Shorack, 2000, Chapter 15)

Due to the bounds on φ_G we have that for any positive real number T there exists a natural number J such that for every $j \geq J$ and every $|s| < T$

$$2^j \left| \ln \varphi_G\left(\frac{s}{2^j}\right) \right| \leq A \frac{|s|^p}{2^{j(p-1)}}.$$

Therefore $\varphi_{G^{\odot}}$ exists and is a characteristic function of a probability measure on \mathbb{R} as an almost uniform limit of characteristic functions of probability measures, proving (iii). The claim of (iv) is a simple corollary to the defining formula, Eq. (4), for $\varphi_{H^{\odot}}$.

□

Remark 4.4 Since $\ln \varphi_G(0) = 1$, the assumption on $\ln \varphi_G$ near zero can be equivalently replaced by the same estimates for $\varphi_G - 1$. This is a direct consequence of the following inequalities valid for all complex numbers a with $|a| < 0.5$,

$$|a|(1 - |a|) \leq |\ln(1 + a)| \leq |a|(1 + |a|).$$

Theorem 4.5

(i) Let $F, G \in \mathcal{P}^{(1)}$. If a centred quadratic stochastic operator \mathbf{Q}_G is weakly asymptotically stable at F with the limit distribution $H^{\odot} \in \mathcal{P}$, then the following equality is satisfied

$$\varphi_{H^{\odot}}(s) = \left(\varphi_{H^{\odot}}\left(\frac{s}{2}\right) \right)^2 \varphi_G(s), \quad s \in \mathbb{R}. \quad (8)$$

(ii) The probability distribution $H \in \mathcal{P}^{(1)}$ is a weak limit of the sequence $(\mathbb{Q}_G^n(F))_{n \in \mathbb{N}}$ for some $F, G \in \mathcal{P}^{(1)}$ if and only if for some characteristic function φ of a probability measure on \mathbb{R} the following equation holds

$$\varphi_H(s) = \left(\varphi_H\left(\frac{s}{2}\right) \right)^2 \varphi(s), \quad s \in \mathbb{R}.$$

Then, for G one can assume $\varphi_G = \varphi$. Moreover, H is a fixed point of such \mathbb{Q}_G , i.e. $\mathbb{Q}_G(H) = H$.

PROOF From the definition of weak stability we have

$$\varphi_{H^{\odot}}(s) = \lim_{n \rightarrow \infty} \left(\varphi_F\left(\frac{s}{2^n}\right) \right)^{2^n} \varphi_{G^{\odot}}(s)$$

for any real s , where $\varphi_{G^{\odot}}$ is defined through Eq. (6). Clearly we then have

$$\begin{aligned} \varphi_G(s) \left(\varphi_{H^{\odot}}\left(\frac{s}{2}\right) \right)^2 &= \lim_{n \rightarrow \infty} \left(\varphi_F\left(\frac{s}{2 \cdot 2^n}\right) \right)^{2 \cdot 2^n} \varphi_G(s) \prod_{j=0}^{n-1} \left(\varphi_G\left(\frac{s}{2 \cdot 2^j}\right) \right) \\ &\stackrel{m=n+1}{=} \lim_{n \rightarrow \infty} \left(\varphi_F\left(\frac{s}{2^m}\right) \right)^{2^m} \varphi_{G^{\odot}}(s) = \varphi_{H^{\odot}}(s). \end{aligned}$$

Hence the first part is proved, as well as Eq. (8), which is equivalent to the following

$$\mathbb{Q}_G(H^{\odot}) = H^{\odot}.$$

The completion of the second part follows directly from the assumption on φ . \square

Remark 4.6 *We notice that Theorem 4.3 is fulfilled by the distributions $F, G \in \mathcal{P}^{(1)}$ of Theorem 4.1, however, $\varepsilon = 1$ is not covered by Theorem 4.3. Indeed, by the implied Strong Law of Large Numbers, F is in the strict domain of attraction to the stable probability distribution concentrated at the mean value $m^{(1)}(F)$. This distribution is stable with any exponent, in particular with exponent 1. Moreover, the zero value of the mean $m^{(1)}(G) = 0$ jointly with the finiteness of the variance $v_G (= m^{(2)}(G))$ implies that $\varphi_G(s) = 1 + vs^2 + o(s^2)$, as $s \rightarrow 0$. Thus the second assumption of Theorem 4.3 is fulfilled with exponent $p = 2$. Obviously, for the particular case of $v = 0$, i.e. of G concentrated at 0, the claims of both theorems follow trivially, whenever F is appropriate. Note, that this corresponds to the lack of any perturbation of the arithmetic mean of the inherited trait. An interesting example of F (due to P. Lévy) where, $\varphi_F(ns) = \varphi_F^n(s)$ for only $n = 2^k$, $k = 1, 2, \dots$, can be found in Feller (1966)'s Chapter 17, Section 3.*

5 Some specific examples

The condition on the logarithm of the kernel's characteristic function in Theorem 4.3 is not an “uncommon” one. Besides distributions G with finite variance, there is a large class of heavy-tailed probability distributions (on \mathbb{R}). Specific subfamilies of this subclass are considered in the propositions below.

Proposition 5.1 *If $G \in \mathcal{P}^{(1)}$ is symmetric and its tails for some constants $C > 0$ and $\varepsilon \in (0, 1)$ satisfy*

$$G(-\infty, -x] = G[x, \infty) \leq Cx^{-(1+\varepsilon)}, \text{ for all } x > 0,$$

then, possibly with another constant $C > 0$, for every $s \in \mathbb{R}$ we have

$$|\ln \varphi_G(s)| < C|s|^{1+\varepsilon},$$

or, equivalently, that $|1 - \varphi_G(s)| < C|s|^{1+\varepsilon}$. In particular, the mean value of G exists (and equals 0).

PROOF Due to the symmetry of G it suffices to consider positive $s > 0$. Moreover, for every $A > 0$ we have

$$\begin{aligned}
|1 - \varphi_G(s)| &= \left| 1 - \int_{\mathbb{R}} e^{isx} dG(x) \right| = \left| \int_{\mathbb{R}} (1 - \cos(sx)) dG(x) \right| \\
&\leq 2 \int_{(0,A)} \frac{s^2 x^2}{2} dG(x) + 4 \int_{[A,\infty)} dG(x).
\end{aligned}$$

By integration of the first term by parts, for $A = \pi/s$ we obtain

$$\begin{aligned}
I &:= 2 \int_{(0,A)} \frac{s^2 x^2}{2} dG(x) = -2 \frac{s^2 A^2}{2} (1 - G(A)) + 2s^2 \int_{(0,A)} xG([x, \infty)) dx \\
&\leq 2s^2 \int_{(0,A)} C|x|^{-\varepsilon} dx = 2C\pi^{1-\varepsilon} s^{1+\varepsilon}.
\end{aligned}$$

For the second term, again for $A = \pi/s$ we have

$$II := 4 \int_{[A,\infty)} dG(x) = 4G([A, \infty)) \leq 4C\pi^{-(1+\varepsilon)} s^{1+\varepsilon}.$$

□

Let us note, that Markov's inequality implies the assumed estimates of tails from finiteness of the absolute moment of order $p = 1 + \varepsilon$, since for positive $x > 0$ we have

$$G(-\infty, -x] + G([x, \infty)) \leq \int_{\mathbb{R}} |u|^p dG(u) / x^p.$$

Next, let us point at the a stronger claim of the asymptotic behaviour of the characteristic function near the origin based on the theory of stability, obviously under stronger assumptions. For practical modelling of real events we present the following corollary to the second part in Borovkov (1972)'s proof of his Theorem 5 (Chapter 7, Section 4).

Corollary 5.2 *If $G \in \mathcal{P}^{(1)}$ with mean value 0 possesses tails such that for some constants $C > 0$ and $\varepsilon \in (0, 1)$ when $x \rightarrow \infty$ behave as*

$$G(-\infty, -x] = Cx^{-(1+\varepsilon)} + o\left(x^{-(1+\varepsilon)}\right), \quad G[x, \infty) = Cx^{-(1+\varepsilon)} + o\left(x^{-(1+\varepsilon)}\right) \quad (9)$$

then

$$\varphi_G(s) - 1 = -2Cc(\varepsilon)|s|^{1+\varepsilon} + o(|s|^{1+\varepsilon}), \text{ as } s \rightarrow 0,$$

$$\text{where } c(\varepsilon) = \frac{(1+\varepsilon)\Gamma(1-\varepsilon)\sin(\varepsilon\frac{\pi}{2})}{\varepsilon}.$$

Similar claims will hold for negative ε , but we are not concerned with this situation. However, the case of $\varepsilon = 0$ is important due to the fact, that functions with such tails are in the domain of attraction to stable probability distributions with characteristic exponent equal to 1 and not concentrated at a single point. An easily checked set of sufficient conditions is given as follows (see again Theorem 5, Chapter 7, Section 4 of Borovkov, 1972; Feller, 1966).

Corollary 5.3 *Under the assumption of Eq. (9) with $\varepsilon = 0$, the characteristic function φ_G , for a symmetric G , satisfies the following limit behaviour*

$$\lim_{n \rightarrow \infty} \left(\varphi_G\left(\frac{s}{n}\right) \right)^n = \exp(-C\pi|s|), \text{ for all } s \in \mathbb{R}.$$

Corollary 5.4 *Every Cauchy-like probability distribution with density*

$$\frac{dF}{d\lambda}(x) = C(1 + a|x - \mu|^\alpha)^{\frac{2}{\alpha}}, \quad x \in \mathbb{R},$$

is in the domain of attraction to the Cauchy probability distribution. In particular it satisfies the conditions of part (i) of Theorem 4.3.

Corollary 5.5 *Every symmetric probability distribution G satisfying one of the below listed requirements fulfills also the assumptions of part (ii) of Theorem 4.3:*

1. $G(-\infty, x] = \frac{(u(x))^\alpha}{(v(x))^\beta}$, $x > 0$, is a positive increasing function not exceeding $\frac{1}{2}$, where u and v are polynomials of degree l and m , respectively, with $\varepsilon := m \cdot \beta - l \cdot \alpha - 1 \in (0, 1)$;
2. $G\{\mathbb{Z}\} = 1$ and $F\{j\} = \frac{(u(j))^\alpha}{(v(j))^\beta}$, $j > 0$, where u and v are positive on \mathbb{Z}_+ polynomials of degrees l and m respectively, with $\varepsilon := m \cdot \beta - l \cdot \alpha - 2 \in (0, 1)$;
3. G is a stable probability distribution of characteristic exponent $1 + \varepsilon$, i.e. with characteristic function $\varphi_G(s) = \exp(-|s|^{1+\varepsilon})$, $s \in \mathbb{R}$.

Thus, in particular, a discrete random variable Y with values in $\mathbb{Z} \setminus \{0\}$ with probabilities

$$P(X = k) = C \frac{1}{|k|^{2+\varepsilon}} \text{ for } k \neq 0, \text{ where } \varepsilon > 0.$$

can serve as a model of perturbation of the inherited trait. The generated QSO \mathbb{Q}_G is stable at every F satisfying the requirements of Theorem 4.1.

Remark 5.6 *Notice that nowhere in this work do we require that the limit of the operator is unique, only that it is to exist for a seed distribution F satisfying the conditions of Theorems 4.1 or 4.3. We also present weak convergence results and we suspect that in many practical cases it will not be possible to obtain strong convergence (L^1) results. It is very plausible that by the law of large numbers we will observe convergence to a Dirac δ . This is an important situation as it indicates fixation of a population and weak convergence handles it perfectly well. However strong convergence will not detect this stabilization. All the iterates of the operator can produce smooth densities hence we will observe an L^1 distance of 2 between all of the iterates and the final measure.*

6 Simulation algorithm

Bartoszek and Pułka (2013a) discussed how simulating quadratic stochastic operators acting on ℓ^1 differs from simulating the trajectory of a Markov linear operator. In the L^1 case we can employ the same procedure to simulate a population behaving according to a kernel quadratic stochastic operator acting on $L^1 \times L^1$. We describe it in Algorithm 1 and Fig. 1 presents histograms from an example run. For the simulations presented in Fig. 1 we considered the normal density kernel with variance 0.5, i.e.

$$q(x, y, z) = \frac{1}{\sqrt{\pi}} \exp(-(z - (x + y)/2)^2) \equiv g(x - \frac{x + y}{2}). \quad (10)$$

One can directly verify that any normal distribution with unit variance, $\mathcal{N}(\mu, 1)$, is a fixed point of the QSO.

6.1 Drawing from $\mathbb{Q}_G^n(F)$

One of the problems of drawing from the laws of iterates of quadratic stochastic operators is that one needs to calculate the distribution function of the n -th iterate.

Algorithm 1 Simulating $\mathbb{Q}(g)$

Draw K independent individuals according to the law of g and call them P_0
for $i = 1$ to n **do**
 $P_i := \emptyset$
 for $j = 1$ to K **do**
 Draw a pair (x_j, y_j) of individuals from population P_{i-1}
 Draw an individual z_j according to the law of $\mathbf{Q}(\delta x_j, \delta y_j) = q(x_j, y_j, \cdot)$
 $P_i = P_i \cup \{z_j\}$
 end for
end for
return P_0, P_1, \dots, P_n

From our experience calculating this directly will result in numerical errors and lengthy calculations times. However the proof of Theorem 4.1 gives an immediate procedure to draw values from the law of a centred kernel QSO, $\mathbb{Q}_G^n(F)$. Namely if we want a value distributed according to the law of $\mathbb{Q}_G^n(F)$ we just need to draw an appropriate (exponential in terms of n) amount of independent random variables distributed according to the laws of F and G . We describe this in Algorithm 2. However the procedure described in Algorithm 2 is naïve in the sense that

Algorithm 2 Naïve drawing from $\mathbb{Q}_G^n(F)$

Draw 2^n random values from the law of F and denote this set $\{X_1, \dots, X_{2^n}\}$.
 $X^{(n)} = \frac{1}{2^n} (X_1 + \dots + X_{2^n})$
for $j = 0$ to $n - 1$ **do**
 Draw 2^j random values from the law of G and denote this set $\{Y_0, \dots, Y_{2^j}\}$.
 $U_j := \frac{1}{2^j} (Y_0 + \dots + Y_{2^j-1})$.
end for
return $X^{(n)} + \sum_{j=0}^{n-1} U_j$

it is exponential in terms of the iteration number. Notice that in the proof of Theorem 4.1 we have that if we assume that F has a finite variance, equalling v_F , $X^{(n)}$ tends almost surely to m and U_j tends almost surely to 0. We should expect these convergences to be rather fast as both random variables are a sum of an exponential number of i.i.d. random variables. We now write

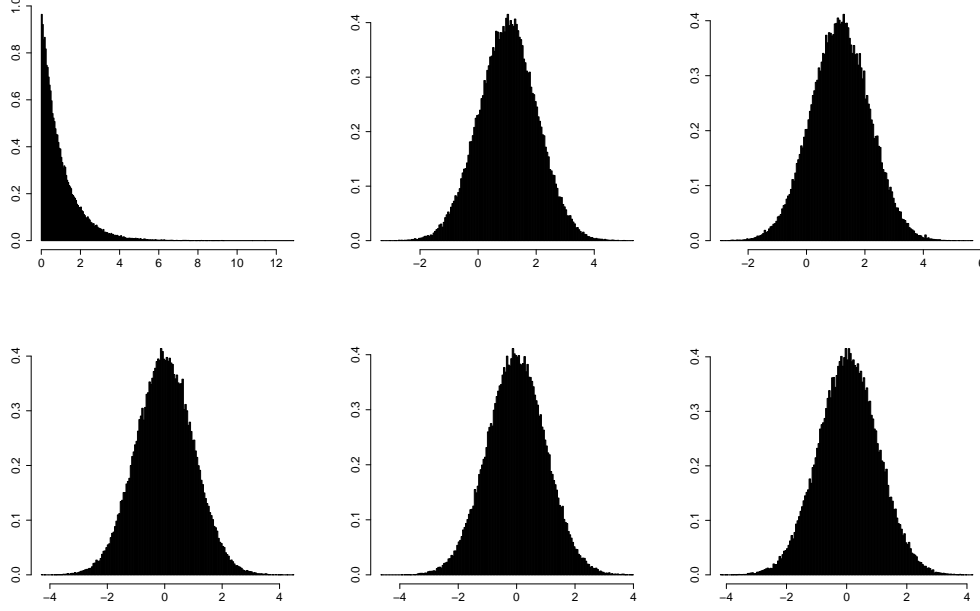


Figure 1: Simulation by Algorithm 1 of a population evolving according to the kernel quadratic stochastic operator $\mathbf{Q}(\cdot, \cdot)$ with $q(x, y, z)$ given by Eq. (10). We assumed $K = 10000$ and present histograms for iterations $n = 1, 100, 500$ (left to right). Top row: initial population drawn from the exponential distribution with rate 1, bottom row: initial population drawn from the standard normal distribution (fixed point of \mathbf{Q}). We can see the mean preserving property, the sample averages are from left to right, top row: 1.001, 1.032, 1.156; bottom row: 0.004, -0.009 and 0.057. Simulations done in R (R Core Team, 2013).

$$H^{(n)} = m + \sum_{j=0}^{N-1} U_j + \varepsilon_{N,n} \equiv m + Y^{(N)} + \varepsilon_{N,n},$$

where $N \leq n$ and $\varepsilon_{N,n}$ is the deviation of $X^{(n)}$ from m and the tail of U_j s. Using Chebyshev we can control the probability that both terms will not exceed a certain value. Remembering that v_G is the variance associated with G we have

$$P(|X^{(n)} - m| > \delta/2) \leq \frac{4v_F}{\delta^2} 2^{-n}$$

and

$$P(|\sum_{j=N}^{n-1} U_j| > \delta/2) \leq \frac{4v_G}{2\delta^2} 2^{-N} (1 - 2^{-n}).$$

Now obviously

$$P(|X^{(n)} - m| > \delta/2) \leq P(|X^{(N)} - m| > \delta/2) \leq \frac{4v_F}{\delta^2} 2^{-N}$$

so for a given δ we can choose N large enough so that the probability, α , of drawing a value which is “off the correct distribution” by more than δ is as small as we desire. Namely we have

$$N(\alpha, \delta, n) \leq \log \frac{4 \max(v_F, v_G(1 - 2^{-n})/2)}{\delta^2 \alpha} = \max(\log \frac{4v_F}{\delta^2 \alpha}, \log \frac{4v_G(1 - 2^{-n})}{\delta^2 \alpha})$$

and if we are interested in drawing from $G^{(\infty)}$

$$N(\alpha, \delta, \infty) \leq \log \frac{4 \max(v_F, v_G/2)}{\delta^2 \alpha}.$$

We describe this modification in Algorithm 3. It still remains an open question

Algorithm 3 Approximate drawing from $\mathbb{Q}_G^n(F)$

```

 $N(\alpha, \delta, n) := \log \frac{4 \max(v_F, v_G(1 - 2^{-n})/2)}{\delta^2 \alpha} + 1$ 
for  $j = 0$  to  $N - 1$  do
    Draw  $2^j$  random values from the law of  $G$  and denote this set  $\{Y_0, \dots, Y_{2^j-1}\}$ .

     $U_j := \frac{1}{2^j} (Y_0 + \dots + Y_{2^j-1})$ .
end for
return  $m + \sum_{j=0}^{N-1} U_j$ 

```

whether drawing 2^N values will be feasible. If we want to draw a population of K individuals then it does not suffice to choose a small α independently of K . This is akin to the multiple testing problem — with K large enough just by chance we will observe an event of probability α . Therefore one way is a “Bonferroni” correction — if on the individual level we want an error with probability α then

in Algorithm 3 we need to take α/K instead of α . We illustrate Algorithm 3 by populations using the same kernel and initial distributions as in Fig. 1. We present the populations' histograms in Fig. 2. For the approximate algorithm we took $\alpha = 0.05$ and $\delta = 0.01$. This resulted in $N = 14$. Each population is of size $K = 10000$ and without the “Bonferroni” style correction sampling of all individuals was instantaneous (about 30s for each on a 1.4GHz AMD Opteron Processor 6274 running Ubuntu 12.04 node of a computational cluster). The crucial tuning parameter is δ . If we choose $\delta = 0.001$ with the same $\alpha = 0.05$ then N rose to 19 and the sampling of the whole population became intolerable. The Bonferroni correction increases N to 23 and 28 respectively for the two values of δ . About 4 hours were needed to simulate the $K = 10000$ individuals population (with $N = 23$) on the same node. However if we compare the histograms of Figs. 1, 2 and 3 we see that there is no need for the correction. In fact the deviations of the mean from its correct value (1 and 0 with the two different seed distributions) are similar in both cases and smaller than in the case of Algorithm 1 for $n = 500$.

6.2 Comparing both algorithms

Both Algorithms 1 and 3 have their advantages and disadvantages. The main advantage of the latter is speed (provided N does not need to be overly large). One just has to simulate values from a univariate G . This is as we approximate the seed distribution by its mean value — indicating that this will work only when n is large, i.e. many iterations have passed and only information on the expectation remains. On the other hand we can control N very precisely as we know F and G . In Section 6.1 we used Chebyshev but for a specific pair of distributions a much better bound will be certainly available. We can expect rapid convergence in distribution as iterations of the operator cause an exponential growth of the number of “independent components” describing the law of $\mathbb{Q}_G^n(F)$. However if N is too large to be practical one can always use a smaller manageable value but then of course the error probabilities will increase. Algorithm 1 allows one to simulate a whole population evolving. This is an advantage if one wants to visualize the evolution. On the other hand and if one is just interested in the law of $\mathbb{Q}_G^n(F)$ or $\mathbb{Q}_G^\infty(F)$ then the need to simulate a whole history can be overly lengthy. This algorithm does not require the drawing of a large number of random variables but has another problem which as we saw in the example simulation caused larger and larger deviations from the true distribution. In a computer simulation we cannot have an infinite population size — only a finite number of individuals. This means that after iterations of mixing more and more dependencies will be appearing in

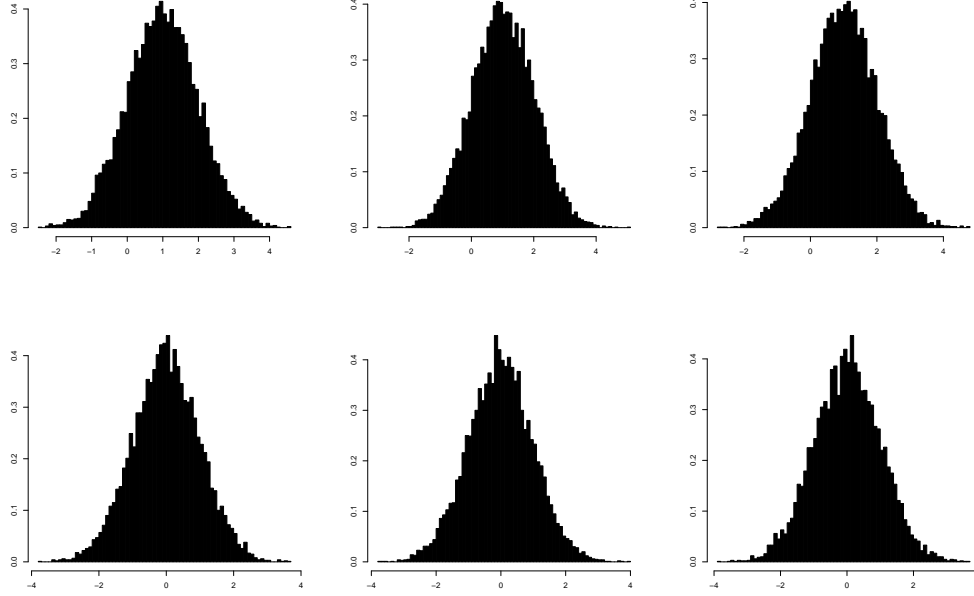


Figure 2: Simulation by Algorithm 3 of the law of kernel quadratic stochastic operator $\mathbf{Q}^n(\cdot, \cdot)$ with $q(x, y, z)$ given by Eq. (10). We assumed $K = 10000$ and present histograms for iterations $n = 1, 100, 500$ (left to right). Top row: initial population drawn from the exponential distribution with rate 1, bottom row: initial population drawn from the standard normal distribution (fixed point of \mathbf{Q}). We can see the mean preserving property, the sample averages are from left to right, top row: 0.989, 1.018, 0.999; bottom row: -0.004 , -0.013 and 0.014 . The top left graph is of course completely wrong as the approximate algorithm has no knowledge of the initial exponential distribution, it uses only its mean and variance. Simulations done in R.

the population — something which our theory at the moment does not account for. In fact when we look at the simulation results presented in Fig. 1 we can see that the population average is slowly deviating from 0 — a consequence of the dependencies due to finite sample size. One can actually think that all of the above issues, especially the exponential number of “independent components” illustrate or rather characterize the complexity of the structure of quadratic stochastic operators. Fortunately one can start quantifying this complexity as we did with the

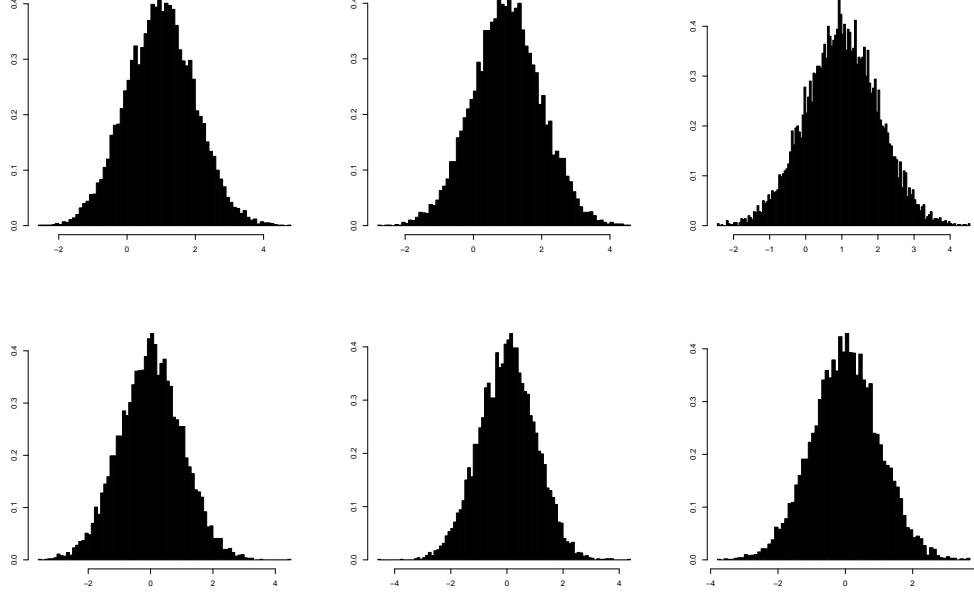


Figure 3: Simulation by Algorithm 3 of the law of kernel quadratic stochastic operator $\mathbf{Q}^n(\cdot, \cdot)$ with $q(x, y, z)$ given by Eq. (10) with the “Bonferroni” type correction. We assumed $K = 10000$ and present histograms for iterations $n = 1, 100, 500$ (left to right). Top row: initial population drawn from the exponential distribution with rate 1, bottom row: initial population drawn from the standard normal distribution (fixed point of \mathbf{Q}). We can see the mean preserving property, the sample averages are from left to right, top row: 0.980, 0.979, 1.003; bottom row: 0.010, 0.017 and 0.007. The top left graph is of course completely wrong as the approximate algorithm has no knowledge of the initial exponential distribution, it uses only its mean and variance. Simulations done in R.

Chebyshev bound. All simulations actually indicate rapid convergence for “decent” F and G distributions. On the other hand we restricted ourselves to a very specific class — centred kernel quadratic stochastic operators. In the full set of quadratic stochastic operators we should expect many more interesting dynamics.

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